Derived annular Khovanov-Rozansky invariants - part 2
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III Categorification

III.1. Soergel bimodules. Fix $n \geq 0$.

- $R := \mathbb{C}[x_1, \dots, x_n]$ $\deg(x_i) = 2$
- R^{s_i} := invariants under $x_i \leftrightarrow x_{i+1}$ $= \mathbb{C}[x_1, \dots, x_i + x_{i+1}, x_i x_{i+1}, \dots, x_n]$
- $B_i := R \otimes_{R^{s_i}} R(1)$ ← grading shift down

$SBim_n$:= full subcat of graded R-R-bimodules
- containing R, B_i $1 \leq i \leq n-1$
- closed under $\otimes_R$, $\oplus$, $\ominus$, grading shift.

– a graded, $\mathbb{C}$-linear (not abelian), monoidal category

$K_0(SBim_n) \cong H_n$

Remark $\boxtimes := \otimes_{\mathbb{C}}$ $SBim_n \times SBim_m \to SBim_{m+n}$, $SSBim_n \times SSBim_m \to SSBim_{m+n}$ $(W_1, W_2) \overset{\boxtimes}{\mapsto} W_1 W_2$

turns $\bigoplus_{n \geq 0} SBim_n$ and $\bigoplus_{n\geq 0} SSBim_n$ into monoidal 2-categories.

III.2 Rouquier complexes and link homology

$T_i := (0 \longrightarrow \underline{B_i} \longrightarrow R(1))$, $1 \otimes 1 \longmapsto 1$

$T_i^{-1} := (R(-1) \longrightarrow \underline{B_i} \longrightarrow 0)$, $1 \longmapsto x_i \otimes 1 - 1 \otimes x_{i+1}$

Thm (Rouquier) The complexes constructed from T_i, T_i^{-1} via $\otimes_R$ satisfy the braid relations up to homotopy equiv.

Remark: $K^b(SBim)$ is an example of a braided monoidal 2-category. one of the very few genuinely 4-categorical algebraic structures known

Thm (Khovanov)

$\{\text{links}\} \xrightarrow{L = \hat{\beta}} \{\text{braids}\} \longrightarrow K^b(SBim_n) \xrightarrow{HH_\bullet} \xrightarrow{H_\bullet} gr^{\mathbb{Z}^3} Vect_{\mathbb{C}}$

$L \mapsto \beta = \sigma_{i_1} \cdots \sigma_{i_k} \mapsto T_{i_1} \otimes \cdots \otimes T_{i_k} \longmapsto HHH(L)$

computes the triply-graded Khovanov-Rozansky homology and categorifies the HOMFLYPT invariant.

IV Categorical traces

IV.1: Vertical trace

"A k-linear category is the same thing as a k-algebra with a collection of orthog. idempotents"

$$HH_0(\mathcal{C}) := \bigoplus_{c \in Ob(\mathcal{C})} End(c) \Big/ FG = GF \text{ for } c_1 \underset{G}{\overset{F}{\rightleftarrows}} c_2$$

If $\mathcal{C}$ is k-linear monoidal, then $HH_0(\mathcal{C})$ inherits an associative multiplication.

Thm (Elias-Lauda)

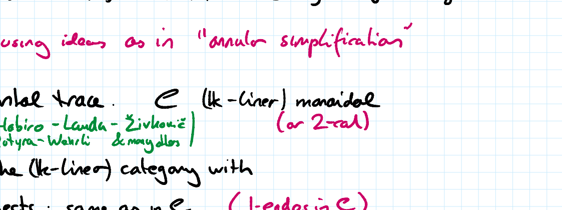
$$HH_0(SBim_n) \cong R \rtimes \mathbb{C}[S_n] \text{ as gr. } \mathbb{C}\text{-algebras}$$

Proof e.g. using ideas as in "annular simplification"

IV.2 Horizontal trace $\mathcal{C}$ (k-linear) monoidal (or 2-cat)

(Beliakova-Habiro-Lauda-Živković, Beliakova-Putyra-Wehrli & many others)

$Tr_0(\mathcal{C})$ = the (k-linear) category with
- objects: same as in $\mathcal{C}$ (1-endos in $\mathcal{C}$)
- $Hom_{Tr_0}(c_1, c_2) := span(c_1 \otimes c \xrightarrow{F} c \otimes c_2)/\sim$

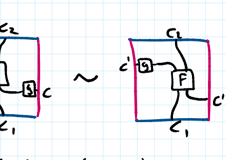


- composition: stacking (uses $\otimes$)

e.g. $Tr_0(\text{tangles in } I^3) = \{\text{Links in } S^1 \times I^2\}$

$Tr_0(\text{braids})$ = positive annular links

Remarks:
- if $\mathcal{C}$ has right-duals: $Tr_0(c_1 \otimes c_2) \cong Tr_0(c_2 \otimes c_1)$
- if $\mathcal{C}$ has left-duals, then $Tr_0 : \mathcal{C} \to Tr_0(\mathcal{C})$ is initial among trace-like functors out of $\mathcal{C}$
- $End_{Tr_0(\mathcal{C})}(1_{\mathcal{C}}) \cong HH_0(\mathcal{C})$



IV.3 Annular link homology à la Queffelec-Rose-Sartori

$\left\{\begin{matrix}\text{annular braid} \\ \text{closures } \hat{\beta}\end{matrix}\right\} \longrightarrow K^b(SBim_n) \longrightarrow K^b(Tr_0(SBim_n)) \xrightarrow{\text{optional}} K^b(KarTr_0(SBim_n))$

$\beta = \sigma_{i_1} \cdots \sigma_{i_k} \mapsto T_{i_1} \otimes \cdots \otimes T_{i_k} \longmapsto AKhR(\hat{\beta})$

QRS describe this as a chain complex of "colored concentric circles" & rotation foams between them

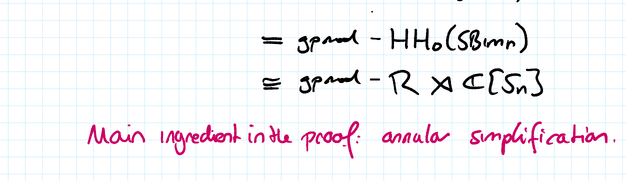
Thm (Gorsky-W) $Hom_{KarTr_0(SBim_n)}(Tr_0(1), -)$

gives an equivalence.

$$KarTr_0(SBim_n) \simeq gpmod\text{-}End_{KarTr_0(SBim_n)}(Tr_0(1))$$
$$= gpmod - HH_0(SBim_n)$$
$$\cong gpmod - R \rtimes \mathbb{C}[S_n]$$

Main ingredient in the proof: annular simplification.

Corollary: $\bigoplus_{n \geq 0} KarTr_0(SBim_n)$ is equivalent to the free symmetric monoidal $\mathbb{C}$-linear, graded, Karoubian category generated by a single object E and an endomorphism $t : E \to E$ of degree 2.



Indecomposables: Schur functors $S^\lambda := S^\lambda(E)$ & their grading shifts

This is the most straightforward (nontrivial) categorification of Λ_q

Applications:
- set $E = \mathbb{C}[X]/X^2$ (≅ vector rep of $U(sl_2)$)
  - $t = 0 \Rightarrow$ annular Khovanov homology (Grigsby-Licata-Wehrli et al.)
  - $t = X \Rightarrow$ Khovanov homology
  - switching t on $\Rightarrow$ AKh → Kh spectral seq.
- set $E = \mathbb{C}[X]/X^N$ (≅ vector rep of $U(gl_N)$)
  - $t = X \Rightarrow$ Khovanov-Rozansky gl_N homology
  - $t = 0 \Rightarrow$ Queffelec-Rose annular gl_N homology
- Hochschild cohomology of $B \in SBim_n$:

$$HH^i(B) = Hom_{KarTr_0(SBim_n)}(S^{\square} \otimes S^{\square}, Tr_0(B))$$

IV.4 Issues

Example 1:

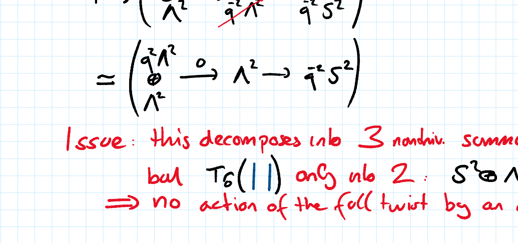
$\overset{\text{braid}}{\mapsto}^{R} (q \, \underline{\square} \to q^{-1} \, \square \to q^{-2} \uparrow\uparrow)$

$\overset{Tr_0}{\mapsto} \left(\begin{matrix} q^2\Lambda^2 \\ \oplus \\ \Lambda^2 \end{matrix} \xrightarrow{0} \begin{matrix}\Lambda^2 \\ q^{-2}\Lambda^2\end{matrix} \to \begin{matrix} q^{-2}\Lambda^2 \\ q^{-2}S^2 \end{matrix}\right)$

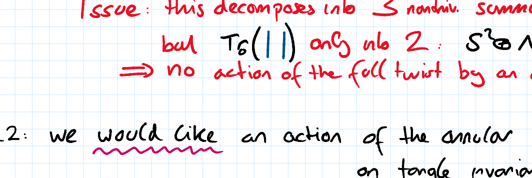
$\simeq \left(\begin{matrix} q^2\Lambda^2 \\ \oplus \\ \Lambda^2 \end{matrix} \xrightarrow{0} \Lambda^2 \to q^{-2}S^2\right)$

Issue: this decomposes into 3 nontriv. summands but $Tr_0(||)$ only into 2: $S^2 \oplus \Lambda^2$
⇒ no action of the full twist by an auto-equiv.

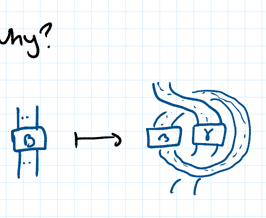
Example 2: we would like an action of the annular invariants on tangle invariants



Issue: $\beta = \gamma = |$ $\hat{\beta} = \bigcirc$



Why?



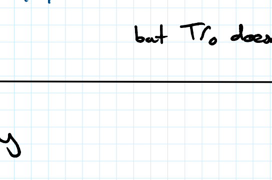
is trace-like only up to homotopy

but Tr_0 doesn't remember homotopies

V Derived story

V.1 Derived traces

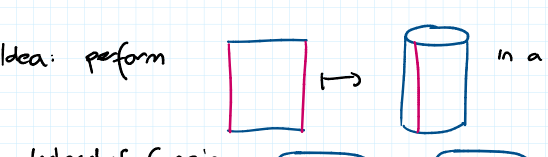
Idea: perform



in a derived way

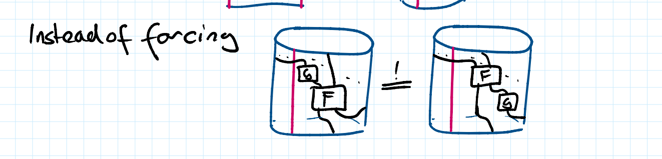
Instead of forcing

we introduce a formal homotopy between them



we get the old relation in homology

this leads to new closed morphisms, which we want to be exact.
⇒ adjoin higher homotopies:

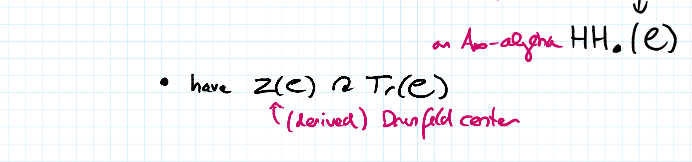


and so on...

differential ∂: alternating sum of erasing "bars"

$\mathcal{B}(\mathcal{C}) = span\langle \; \rangle$ is just the 2-sided bar-complex of $\mathcal{C}$.

$Tr(\mathcal{C})$ = the dg category with
- objects: same as in $\mathcal{C}$
- $Hom_{Tr}(c_1, c_2) :=$



- composition via shuffle product and $\otimes$ in $\mathcal{C}$.

Remarks:
- $Tr(\mathcal{C}) = \mathcal{B}(\mathcal{C}) \boxtimes_{\mathcal{C}_1 \boxtimes \mathcal{C}_4^{op}} \mathcal{C}_1 \boxtimes \mathcal{C}_2 \,{}^{\mathcal{C}}\!\mathcal{C}_3 \boxtimes \mathcal{C}_4$
- similar for dg monoidal $\mathcal{C}$.
- $End_{Tr}(1_{\mathcal{C}}) \cong \mathcal{B}(\mathcal{C}) \boxtimes_{\mathcal{C} \boxtimes \mathcal{C}^{op}} \mathcal{C} \cong HC_\bullet(\mathcal{C})$ ← a dg algebra for monoidal $\mathcal{C}$ → an A_∞-algebra $HH_\bullet(\mathcal{C})$
- have $Z(\mathcal{C}) \curvearrowright Tr(\mathcal{C})$ ← (derived) Drinfeld center
- $Tr(\mathcal{C})$ is usually not idempotent complete nor triangulated
  formally complete ⇒ $\widetilde{Tr}(\mathcal{C})$

V.2 Hochschild homology of SBim

Thm (Libedinsky-Williamson)

$Ch^b(SBim_n)$:= bounded chain cx in $SBim_n$ (monoidal dg cat. of)

has a semi-orthogonal decomposition on Rouquier cx's T_w of pos. permutation braids, $w \in S_n$

there are no morphisms going down in the Bruhat order/hty

Thm (Gorsky-Hogencamp-W)

$$HC_\bullet(SBim_n) \simeq HC_\bullet(Ch^b(SBim_n))$$
$$\simeq \bigoplus_{w \in S_n} HC_\bullet(End_{Ch^b(SBim_n)}(T_w)) \quad \text{(s.-o. decomp.)}$$
$$\simeq \bigoplus_{w \in S_n} HC_\bullet(End(T_1))$$
$$= \bigoplus_{w \in S_n} HC_\bullet(R)$$

$\Rightarrow HH_\bullet(SBim_n) \cong HH_\bullet(R) \rtimes \mathbb{C}[S_n]$

↑ as A_∞ algebras ↑ μ_2 is what you think it is

formal?

$$\cong \mathbb{C}[x_1 \dots x_n, \theta_1, \dots, \theta_n] \rtimes \mathbb{C}[S_n]$$

$\deg: (2,0)$ $\deg: (2,-1)$

Remark: this generalizes to Coxeter groups

$HH_\bullet(SBim_{(W,\mathfrak{h})}) \cong HH_\bullet(Sym(\mathfrak{h}^*)) \rtimes \mathbb{C}W$

V.3 Trace of $SBim_n$

Thm (GHW) $\widetilde{Tr}(SBim_n)$ is generated by $Tr(\mathbb{1})$

$Hom_{\widetilde{Tr}}(Tr(\mathbb{1}), -)$ establishes a quasi-equivalence

$\widetilde{Tr}(SBim_n) \simeq$ perfect right A_∞-modules / $End_{Tr}(Tr(\mathbb{1}))$

$\overset{\text{Thm}}{=} Perf(\mathbb{C}[x_1 \dots x_n, \theta_1, \dots, \theta_n] \rtimes \mathbb{C}[S_n])$

"dots" "dot rotation homotopies" "concentric circle permutations"

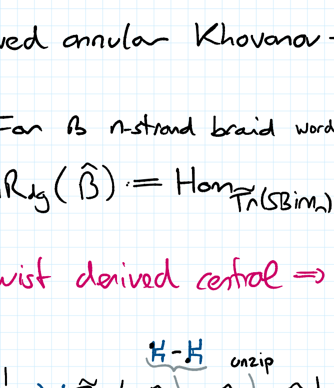
V.4 Derived annular Khovanov-Rozansky invariants

Definition: For β n-strand braid word:

$$AKhR_{dg}(\hat{\beta}) := Hom_{\widetilde{Tr}(SBim_n)}(Tr(\mathbb{1}), Tr(T_\beta))$$

Full twist derived central ⇒ acts on $AKhR_{dg}$

Example 1



two components! $FT(\Lambda^2)$ & $FT(S^2)$

Example 2 gets resolved analogously.