From Link Homology to TQFTs

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Overview

Introduction

Challenge

1. The Witten-Reshetikhin–Turaev (WRT) invariants of 3-manifolds arise from ribbon categories of quantum group representations.
   - For example, as sums of colored Jones polynomials at a root of unity.
2. Much of quantum group representation theory categorifies.
   - Khovanov taught us how to categorify the colored Jones polynomials.

Challenge (following Crane-Frenkel, Khovanov)

Use **categorification** to build invariants that are:

- algebraically computable, e.g. in terms of link homology
- useful for smooth 4-manifold topology

Approach

1. Understand conceptual background of WRT invariants via a family of extended topological quantum field theories (TQFTs).
   - TQFTs are determined by algebro-categorical **local data**.
2. Categorify the local data.
   - Categorification is non-deterministic, explore choices!
3. Construct categorified TQFTs from categorified local data.
   - Results will depend on categorification choices.
   - Categorification may circumvent finiteness constraints.

Talk today: recent progress

1. First approximation satisfies the wish list:
   - computable in terms of link homology
   - useful for smooth 4-manifold topology
2. Second generation invariants show hints of WRT categorification.

Link homology and TQFT background

Link homology

$$
\begin{array}{ccc}
\left\{ \begin{array}{c} \text{links diagrams in } \mathbb{R}^2 \\ \text{movies of diagrams/m. moves} \end{array} \right\} & \xrightarrow{\quad H \quad} & K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}) \\
\updownarrow \cong & & \downarrow \chi_q \\
\left\{ \begin{array}{c} \text{links embedded in } \mathbb{R}^3 \\ \text{cobordisms in } \mathbb{R}^3 \times I/\text{isotopy} \end{array} \right\} & \xrightarrow{\quad P \quad} & \mathbb{Z}[q^{\pm 1}]
\end{array}
$$

Link homology H via generators and relations (Khovanov 99):

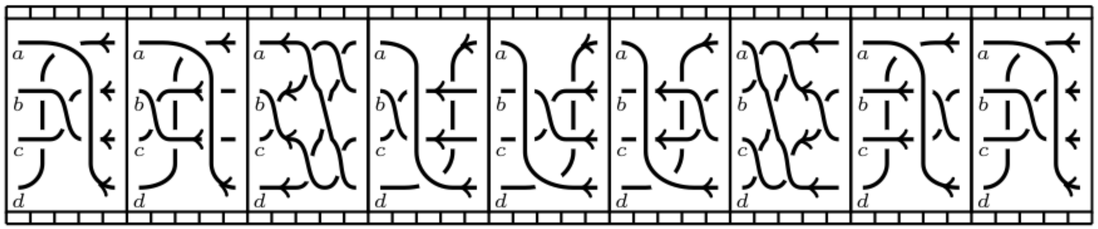
- data: chain complex for each link diagram
- data: a chain map for every elementary movie
- property: satisfying movie moves

Theorem (Ehrig–Tubbenhauer–W. 2017)

General linear link homology yields such a functor, via Robert–Wagner combinatorial approach to $\mathfrak{gl}_N$ Khovanov–Rozansky homology.

Link homology

E.g. these chain maps should be homotopic to the identity:



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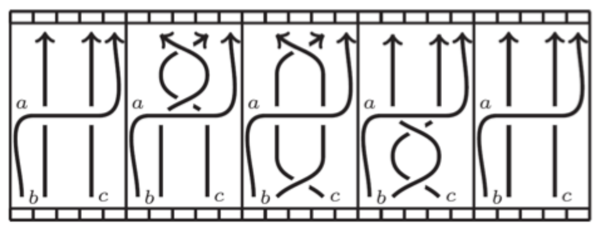

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- property: satisfying movie moves

Theorem (Ehrig–Tubbenhauer–W. 2017)

General linear link homology yields such a functor, via Robert–Wagner combinatorial approach to $\mathfrak{gl}_N$ Khovanov–Rozansky homology.

TQFT

A TQFT is an assignment:

$$\text{spacetimes} \rightarrow \text{algebro-categorical structures}$$

compatible with gluing:

- for a fixed spatial dimension n and tangential structure
- extension up: functorial under some $(n+1)$-dimensional bordisms
- extension down: higher algebraic structures for lower dimensional manifolds needed for gluing
- computation of manifold invariants via cut-and-paste techniques

Terminology

A TQFT is **local** if value on the point determines all values via gluing.

Today: all manifolds smooth, oriented.

Best case scenario for local $(n+1)$-dimensional TQFT

manifold dimension	type of value on closed manifolds
$n+1$	scalars
n	vector spaces
$n-1$	1-categories
$\vdots$	$\vdots$
$n-k$	k-categories
$\vdots$	$\vdots$
0	n-categories

Ignored aspects:

- Ambient symmetric monoidal higher category
- Values on manifolds with boundary
- Dualizability, incl. finiteness
- Pivotality (tangential structure requirements)

Example: Turaev–Viro type local $(2+1)$d TQFT

manifold dimension	type of value on closed manifolds	
3	scalars	Turaev–Viro (TV) invariants
2	vector spaces	skein modules of 2d $\mathcal{C}$-diagrams
1	certain categories	skein categories of 2d $\mathcal{C}$-diagrams
0	certain 2-categories	pt $\mapsto$ spherical fusion category $\mathcal{C}$

No details here, but see e.g.:

- Turaev–Viro, Ocneanu 92-94: state sum 3-manifold invariant from triangulation and 6j-symbols
- Barrett-Westbury 96-99: from spherical fusion category
- Roberts 93, Walker: relation to skein theory
- Balsam–Kirillov 10: as 1-2-3 extended TQFT
- Douglas–Schommer-Pries–Snyder 13: local framed version via cobordism hypothesis

Example: Crane–Yetter type local $(3+1)$d TQFT

manifold dimension	type of value on closed manifolds	
4	scalars	Crane–Yetter (CY) invariants
3	vector spaces	skein modules of 3d $\mathcal{C}$-diagrams
2	certain categories	skein categories of 3d $\mathcal{C}$-diagrams
1	certain 2-categories	
0	certain 3-categories	pt $\mapsto$ ribbon fusion category $\mathcal{C}$

No details here, but see e.g.:

- Crane–Yetter 93: state sum 4-manifold invariant from triangulation and 15j-symbols
- Crane–Kauffman–Yetter 94: from ribbon fusion category
- Roberts 93, Walker,. . . , Tham 21: relation to skein theory

Caveat

CY of 4-manifolds only depends on Euler characteristic and signature.

Example: Reshetikhin–Turaev type 1-2-3d TQFT

manifold dimension	type of value on closed manifolds
3	scalars
2	vector spaces
1	certain categories

WRT invariants

$S^1 \mapsto$ modular fusion cat $\mathcal{C}$

Caveat

RT may not be local, but instead a boundary theory for CY.
Walker, Freed–Telemann, Johnson-Freyd–Scheimbauer, Haïoun...

Example

1. Surgery presentation of M^3 defines cobordism $W^4 \colon M^3 \to \emptyset$.
2. Evaluate map $CY(W^4)$ on vacuum skein $\emptyset \in CY(M^3)$.
3. Scalar renormalization removes dependence on W^4, yields $RT(M^3)$

Forgetting the braiding yields relationship: $TV(M^3) = |RT(M^3)|^2$.

Takeaways

1. Quantum group representation theory yields a family of TQFT triples
   - local 3d TV
   - local 4d CY
   - 1-2-3d RT
2. Top dimension in TV & CY need finiteness, roots of 1, semisimplicity. Lower dimensions more robust via skein theory
   $\Longrightarrow$ local $(n+\varepsilon)$-dimensional TQFT, maybe partially defined Walker
3. Categorified RT may need categorified CY and categorified TV.

Strategy

Categorification and skein theory should give access to:

- Categorified CY in dimensions ≤ 4.
- Categorified TV in dimensions ≤ 3.

Periodic table of $(n+\varepsilon)$-dimensional TQFTs

Inspired by variations of the cobordism hypothesis Baez–Dolan, Lurie.
By **local data**:

$\mathbb{E}_k \backslash n-k$	0	1	2	$\cdots$
—	sets	categories	2-categories	$\cdots$
$\mathbb{E}_1$	monoids	monoidal cats	monoidal 2-cats	$\cdots$
$\mathbb{E}_2$	comm. monoids	braided cats	braided 2-cats	$\cdots$
$\mathbb{E}_3$	—	sym. mon. cats	sylleptic 2-cats	$\cdots$
$\mathbb{E}_4$	—	—	sym. mon. 2-cats	$\cdots$
$\vdots$	—	—	—	$\ddots$

- Dimension $n =$ (category level $n-k$) + (degree of monoidality k).

Periodic table of $(n + \varepsilon)$-dimensional TQFTs

Inspired by variations of the cobordism hypothesis Baez–Dolan, Lurie.
By **shape of skeins**:

$\mathbb{E}_k \backslash n-k$	0	1	2	$\cdots$
—				$\cdots$
$\mathbb{E}_1$	points in 1d	lines in 2d	surfaces in 3d	$\cdots$
$\mathbb{E}_2$	—	lines in 3d	surfaces in 4d	$\cdots$
$\mathbb{E}_3$	—	—	surfaces in 5d	$\cdots$
$\mathbb{E}_4$	—	—	—	$\cdots$
$\vdots$	—	—	—	$\ddots$

- Dimension $n =$ (category level $n-k$) + (degree of monoidality k).
- Skeins of codimension k in ambient n-manifolds.

Four types of TQFT from Categorification

TQFTs from Categorification

$\mathbb{E}_k \backslash n-k$	linear 1-categories	loc. linear 2-categories
monoidal	TV	Asaeda–Frohman–Kaiser
braided	CY	

1. Asaeda–Frohman 07, Kaiser 09: skein modules of surfaces in 3d

Douglas–Reutter 18: skein modules of foams in 3d

- based on concept of fusion 2-categories
- extends to 4d by state sum
- maybe no oriented exotica detection due to semisimplicity

TQFTs from Categorification

$\mathbb{E}_k \backslash n-k$	linear 1-categories	loc. linear 2-categories	loc. stable $(\infty, 2)$-categories
monoidal	TV	Asaeda–Frohman–Kaiser	[HRW24]
braided	CY	[MWW19]	[LMGRSW24]

1. Asaeda–Frohman 07, Kaiser 09: skein modules of surfaces in 3d
2. Morrison–Walker–W. 19: skein modules of surfaces in 4d
   - based on link homology, e.g. $\mathfrak{gl}_N$ Khovanov–Rozansky homology
3. Liu–Mazel-Gee–Reutter–Stroppel–W. 24:
   - local data for **derived skein modules** of surfaces in 4d
   - based on Rouquier complexes from link homology
4. Hogancamp–Rose–W. 24:
   - prototype **derived skein modules** of surfaces in 3d
   - relation to Rozansky–Willis invariants, stable RT categorification

Asaeda–Frohman–Kaiser type $(3+\varepsilon)$d TQFT

$\mathbb{E}_k \backslash n-k$	loc. linear 2-categories	loc. stable $(\infty, 2)$-categories
monoidal	Asaeda–Frohman–Kaiser	
braided		

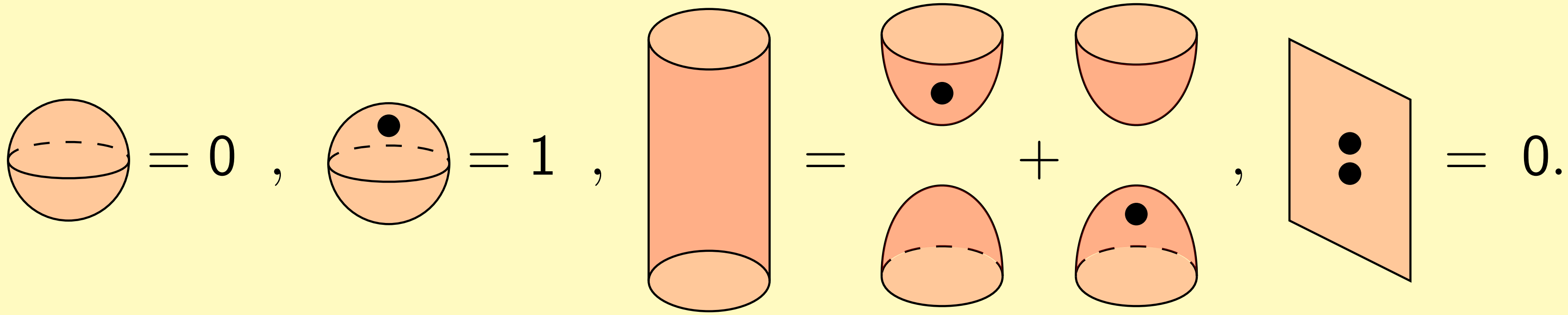
Asaeda–Frohman–Kaiser type $(3+\varepsilon)$d TQFT

Local data

Locally linear monoidal 2-category constructed from e.g.:

- a commutative Frobenius algebra $\equiv$ classical 1-2d TQFT, or
- a closed foam evaluation formula Blanchet 10, Robert–Wagner 17, Kronheimer–Mrowka+Khovanov–Robert 15-18

Example: Khovanov–Bar-Natan skein theory



Asaeda–Frohman–Kaiser type $(3+\varepsilon)$d TQFT

Local data

Locally linear monoidal 2-category constructed from e.g.:

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- a closed foam evaluation formula Blanchet 10, Robert–Wagner 17, Kronheimer–Mrowka+Khovanov–Robert 15-18

- $M^3 \mapsto$ skein module of decorated surfaces/foams in M^3 mod relations
- $\Sigma^2 \mapsto$ linear category with morphism spaces from values on $\Sigma^2 \times I$
  - used for Khovanov homology of links in thickened surfaces Boerner 08
  - categorifies Temperley–Lieb skein module of Σ^2 Queffelec–W. 18

(4 + ε)d TQFT from link homology

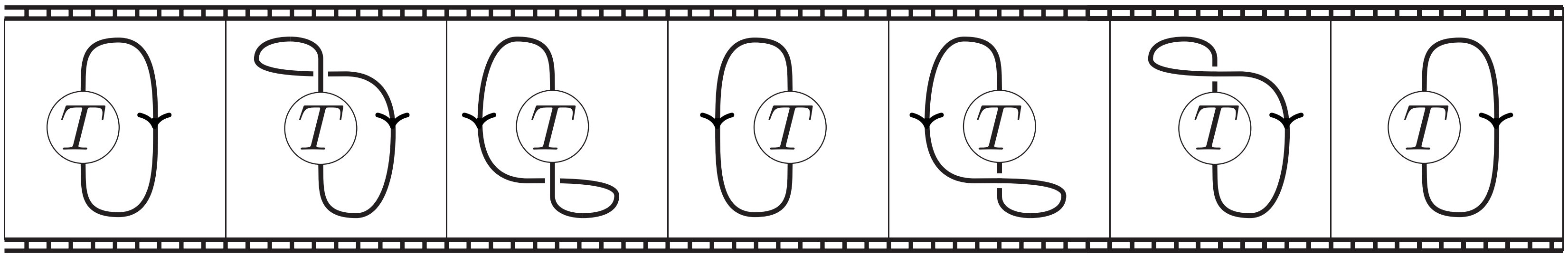
$\mathbb{E}_k \backslash n - k$	loc. linear 2-categories	loc. stable $(\infty, 2)$-categories
monoidal		
braided	[MWW19]	

$(4+\varepsilon)$d TQFT from link homology

Local data (Morrison–Walker–W. 19)

Locally linear braided monoidal 2-category constructed from link homology that is functorial in B^3, $\mathbb{E}_3$-monoidal, and 3-spherical, e.g. $\mathfrak{gl}_N$ homology.

Infinite family of extra sweeparound movie moves:



$(4+\varepsilon)$d TQFT from link homology

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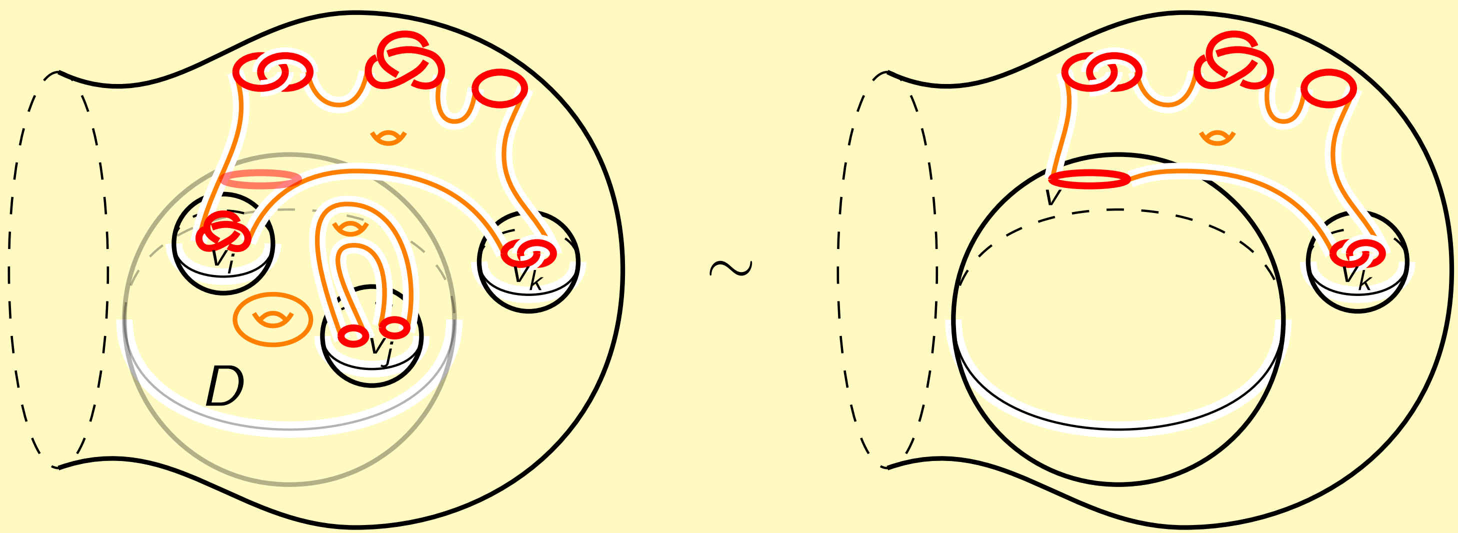
- $W^4 \mapsto$ skein module of decorated surfaces in W^4 mod relations

Skein relations from Morrison–Walker–W. 19



Variation for Floer lasagna modules Chen 22.

$(4+\varepsilon)$d TQFT from link homology

Local data (Morrison–Walker–W. 19)

Locally linear braided monoidal 2-category constructed from link homology that is functorial in B^3, $\mathbb{E}_3$-monoidal, and 3-spherical, e.g. $\mathfrak{gl}_N$ homology.

- $W^4 \mapsto$ skein module of decorated surfaces in W^4 mod relations
- $M^3 \mapsto$ linear category with morphism spaces from values on $M^3 \times I$
- link homology is skein module of B^4 with link as boundary condition

Theorems (Manolescu–Neithalath 20, Manolescu–Walker–W. 22)

Skein modules for $\mathfrak{gl}_N$ homology can be reduced to link homology in S^3 along a handle decomposition of W^4.

Theorem (Hogancamp–Rose–W. 22)

The Khovanov homology skein module for the surgery cobordism of a link is an explicit colimit of colored Khovanov homologies.

Rhymes with surgery description of RT. Computability ✓.

$(4+\varepsilon)$d TQFT from link homology

Theorem (Sullivan–Zhang 24)

The Khovanov skein module of $S^2 \times S^2$ vanishes.

Theorems (Morrison–Walker–W. 24)

- Deformed $\mathfrak{gl}_N$ skein modules as $\bigoplus_i \mathfrak{gl}_{N_i}$ skein modules.
- Genus bounds (Rasmussen s-invariants) for smooth surfaces in W^4.

E.g. Khovanov $\mathfrak{gl}_2$ skein modules deform to Lee $\mathfrak{gl}_1 \oplus \mathfrak{gl}_1$ skein modules.

Theorems (Ren–Willis 24)

- Vanishing results for certain Khovanov skein modules, e.g. high framing knot traces, inherited by embeddings.
- Diagrammatic non-vanishing results.
- Purely algebro-combinatorial detection of exotica!

Applications ✓. But these skein modules do not categorify RT!

$(4+\varepsilon)$d TQFT from link homology

Caveat

Manolescu–Walker–W. 22: Khovanov skein modules can be locally infinite dimensional $\implies$ have no decategorification.

Example: Consider $B^3 \times S^1$ with link $\{4 \text{ points}\} \times S^1$.

- Skein module is HH_0 of linear category associated to $(B^3, \{4 \text{ points}\})$.
- Every rational 4-ended tangle gives an object.
- Their rotation surfaces are linearly independent skeins in degree 0. □

What happened?

Chain complexes for 4-ended tangles fail to decategorify to their Euler characteristic under HH_0. Have taken (link) homology too early.

Remedy

Work on level of chain complexes instead of link homology.
Example recovers Rozansky's Khovanov homology for links in $S^2 \times S^1$.

derived $(4+\varepsilon)$d TQFT via link complexes

$\mathbb{E}_k \backslash n-k$	loc. linear 2-categories	loc. stable $(\infty, 2)$-categories
monoidal		
braided		[LMGRSW24]

derived $(4+\varepsilon)$d TQFT via link complexes

Local data (wanted!)

Locally stable $\mathbb{E}_2$-monoidal $(\infty, 2)$-category constructed from link chain complexes, 4-dualizable in a suitable symmetric monoidal $(\infty, 5)$-category and equipped with $SO(4)$-homotopy fixed point data (pivotality).

Theorem (Liu–Mazel-Gee–Reutter–Stroppel–W. 24)

Chain complexes of type A Soergel bimodules assemble into a locally stable $\mathbb{E}_2$*-monoidal* $(\infty, 2)$*-category with braiding by Rouquier complexes.*

Objects not dualizable. Only braids, no tangles. Triply-graded homology.

Theorem (Dyckerhoff–W. 25 inspired by Kapranov–Schechtman)

Braiding comes from factorizing family of perverse schobers on $\mathrm{Sym}^\bullet(\mathbb{C})$.

Challenges

- Build $\mathfrak{gl}_N$ version, generated by 2-dualizable objects, pivotality.
- Globalize to derived skein modules, β-factorization homology.

derived $(3+\varepsilon)$d TQFT

$\mathbb{E}_k \backslash n-k$	loc. linear 2-categories	loc. stable $(\infty, 2)$-categories
monoidal		[HRW24]
braided		

derived $(3+\varepsilon)$d TQFT

Parallel bordered (sutured) HF package Lipshitz–Ozsvath–Thurston, Zarev, Douglas–Manolescu, Rouquier–Manion?

Want:

- $M^3 \mapsto$ chain complex (derived skein module)
- $\Sigma^2 \mapsto$ dg category with morphism spaces from values on $\Sigma^2 \times I$

Local data

Locally linear monoidal 2-categories as in Asaeda–Frohman–Kaiser TQFT.

$\implies$ skein theory in B^3 (contractible) should be the same (discrete).

Idea

- Every M^3 arises from gluing B^3s along parts of their boundaries.
- Model gluing as derived $\otimes$ over dg category for gluing locus Σ^2.

This is not as circular as it sounds! Same strategy for Σ^2.

derived $(3+\varepsilon)$d TQFT

Theorems (Hogancamp–Rose–W. 24)

For every marked surface Σ^2, there exists a canonically associated dg category that

- has homotopy category $AFK(\Sigma^2)$ and K_0 the TL skein module of Σ^2
- can be computed from any choice of 2d 1-handlebody structure of Σ^2
- tautologically carries a coherent action of $\mathrm{Diff}^+(\Sigma^2)$
- graded hom complexes have locally finite-dimensional cohomology
  - instances of Rozansky–Willis invariants
- hom pairing categorifies the natural hermitian pairings on
  - $\mathrm{TV}(\Sigma^2)$
  - $\mathrm{CY}(\Sigma^2 \times I)$
  - $\mathrm{RT}(\Sigma^2 \cup_{\partial\Sigma} \overline{\Sigma^2})$
- Cooper–Krushkal categorified spin networks form generating objects for completion, orthogonal for symmetrized hom.

Summary

- RT as boundary of CY and square root of TV, guide categ'f'n.
- Lower dimensional layers accessible via (derived) skein theory.
- Start seeing two candidate categorified CY and categorified TV each:

$\mathbb{E}_k \backslash n-k$	linear 1-categories	loc. linear 2-categories	loc. stable $(\infty, 2)$-categories
monoidal	TV	Asaeda–Frohman–Kaiser	[HRW24]
braided	CY	[MWW19]	[LMGRSW24]

- First generation: linear skein theories
  - computability, ready for applications
  - sensitivity
- Second generation: derived skein theories
  - expect better decategorification behavior
  - technically challenging, needs expertise from many directions